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# 4

## VECTORS IN THE PLANE

# APPLICATIONS OF VECTORS

### TERMINOLOGY

airspeed
component
course made good
dot product
ground speed
heading
orthogonal component
parallel component
projection
resolution
resolve
scalar product
speed made good
speed through the water
true course
wind
work



Prior learning

## ALGEBRA OF VECTORS IN THE PLANE

- define and use scalar (dot) product (ACMSM019)
- apply the scalar product to vectors expressed in component form (ACMSM020)
- examine properties of parallel and perpendicular vectors and determine if two vectors are parallel or perpendicular (ACMSM021)
- define and use projections of vectors (ACMSM022)
- solve problems involving displacement, force and velocity involving the above concepts. (ACMSM023)

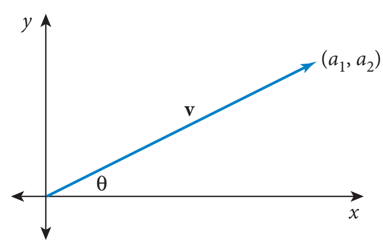
## GEOMETRIC PROOFS USING VECTORS IN THE PLANE

- the diagonals of a parallelogram meet at right angles if and only if it is a rhombus (ACMSM039)
- the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides. (ACMSM041) AC

# 4.01 THE SCALAR PRODUCT

In Chapter 1, you saw that a vector is a quantity with both magnitude and direction.

The vector **v** shown on the right has a magnitude $v$, where $v = |\mathbf{v}| = \sqrt{a_1^2 + a_2^2}$, and makes an angle of $\theta$ with the positive direction of the $x$-axis.




The polar form of **v** is $(r, \theta)$.

The component form of **v** is $(a_1, a_2)$ or $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$.

The unit vector form of **v** is $a_1\mathbf{i} + a_2\mathbf{j}$, where $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.

You already know how to add two vectors and how to multiply a vector by a scalar.

The **scalar product** or **dot product** of two vectors produces a scalar (real number) answer.

### IMPORTANT

For two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ with an angle of $\theta$ between them, the **scalar product** or **dot product** is given by $\mathbf{v}_1 \cdot \mathbf{v}_2 = |\mathbf{v}_1| \times |\mathbf{v}_2| \cos(\theta)$

### Example 1

Find the scalar product of vectors with magnitudes 5 and 6 at an angle of 60° to each other.

#### Solution

Write the formula. $\mathbf{v}_1 \cdot \mathbf{v}_2 = |\mathbf{v}_1| \times |\mathbf{v}_2| \cos(\theta)$

Substitute in the values. $= 5 \times 6 \times \cos(60°)$

Evaluate. $= 15$

The dot product can be positive or negative.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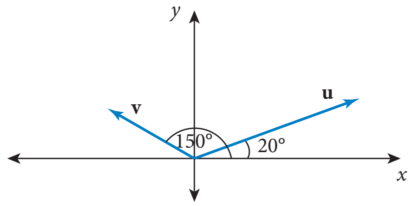
## Example 2

If **u** = (4, 20°) and **v** = (2, 150°), calculate **u** · **v** correct to 2 decimal places.

### Solution

Sketch a diagram.




Find the angle between the vectors. $\theta = 150° - 20° = 130°$

Write the formula. $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \times |\mathbf{v}| \cos(\theta)$

Substitute in the values. $= 4 \times 2 \times \cos(130°)$

Evaluate and round to 2 decimal places. $\approx -5.14$

You can use your CAS calculator to find the dot product of vectors.

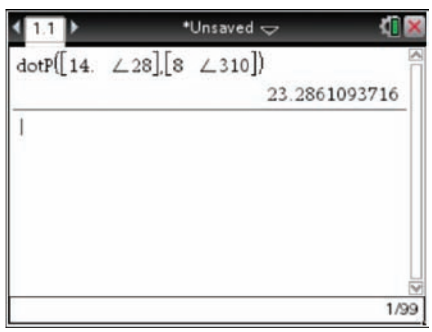
## Example 3

Use your CAS calculator to find (14, 28°) · (8, 310°), correct to 6 significant figures.

### Solution

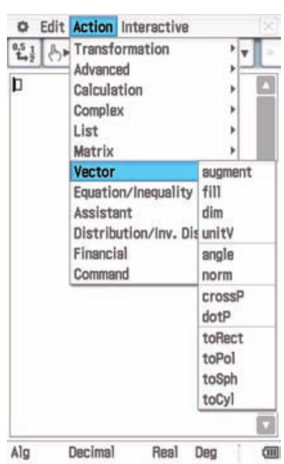
**TI-Nspire CAS**

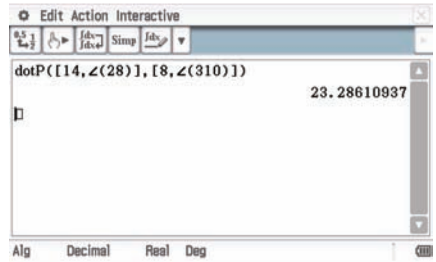
Use a calculator page.
Make sure that your calculator is set to degrees and the calculation mode is set to approximate (or put a decimal point in one of the numbers).
Use menu, 7: Matrix & Vector, C: Vectors and 3: Dot Product to get dotP(). Enter the vectors separated by a comma.




**ClassPad**

Use the Main $\sqrt{\alpha}$ menu.
First set the calculator to degrees (**Deg**) and **Decimal**.
Tap **Action**, then **Vector** and then **dotP**.
Enter the two vectors, separated by a comma.
Tap EXE or press the EXE key. Remember to put each vector in square brackets.
The vectors can be entered in either polar or component form.
To get the angle sign, press Keyboard, tap Math3.




Write the answer to the specified accuracy. $(14, 28°) \cdot (8, 310°) \approx 23.2861$

# EXERCISE 4.01 The scalar product

## Concepts and techniques

1 Example 1 The angle between vectors **a** and **b** is $\theta$. Find $\mathbf{a} \cdot \mathbf{b}$ (correct to 2 decimal places if necessary) for each of the following.

- a $|\mathbf{a}| = 2$, $|\mathbf{b}| = 7$ and $\theta = 30°$
- b $|\mathbf{a}| = 1$, $|\mathbf{b}| = 11$ and $\theta = 60°$
- c $|\mathbf{a}| = 5$, $|\mathbf{b}| = 8$ and $\theta = 45°$
- d $|\mathbf{a}| = 7$, $|\mathbf{b}| = 3$ and $\theta = 115°$
- e $|\mathbf{a}| = 6$, $|\mathbf{b}| = 5$ and $\theta = 90°$
- f $|\mathbf{a}| = 6$, $|\mathbf{b}| = 5$ and $\theta = 0°$
- g $|\mathbf{a}| = 4$, $|\mathbf{b}| = 9$ and $\theta = 184.6°$
- h $|\mathbf{a}| = 12$, $|\mathbf{b}| = 8$ and $\theta = 311.6°$

2 The angle between vectors **v** and **w** is $\theta$. Find $\mathbf{v} \cdot \mathbf{w}$ (correct to 2 decimal places if necessary) for each of the following.

- a $|\mathbf{v}| = 3$, $|\mathbf{w}| = 2$ and $\theta = \frac{\pi}{3}$
- b $|\mathbf{v}| = 4$, $|\mathbf{w}| = 7$ and $\theta = \frac{\pi}{6}$
- c $|\mathbf{v}| = 8$, $|\mathbf{w}| = 5$ and $\theta = \frac{\pi}{4}$
- d $|\mathbf{v}| = 7$, $|\mathbf{w}| = 9$ and $\theta = \frac{3\pi}{4}$
- e $|\mathbf{v}| = 9$, $|\mathbf{w}| = 3$ and $\theta = \frac{\pi}{2}$
- f $|\mathbf{v}| = 9$, $|\mathbf{w}| = 3$ and $\theta = 0$
- g $|\mathbf{v}| = 11$, $|\mathbf{w}| = 7$ and $\theta = \frac{8\pi}{5}$
- h $|\mathbf{v}| = 8$, $|\mathbf{w}| = 15$ and $\theta = \frac{15\pi}{7}$

3 Example 2 Calculate each of the following (correct to 2 decimal places).

- a $(18, 354°) \cdot (14, 157°)$
- b $(10, 279°) \cdot (17, 152°)$
- c $(11, 65°) \cdot (17, 345°)$
- d $(9, 256°) \cdot (9, 260°)$
- e $(10, 307°) \cdot (14, 219°)$
- f $(4, 202°) \cdot (19, 241°)$
- g $(12, 314°) \cdot (7, 140°)$
- h $(8, 236°) \cdot (8, 40°)$

4 Example 3 CAS Find each of the following (correct to 2 decimal places).

- a $(16, 53°) \cdot (18, 259°)$
- b $(12, 204°) \cdot (13, 359°)$
- c $(14, 359°) \cdot (24, 313°)$
- d $(7, 195°) \cdot (4, 35°)$
- e $(10, 125°) \cdot (22, 31°)$
- f $(13, 137°) \cdot (13, 134°)$
- g $(14, 19°) \cdot (4, 154°)$
- h $(6, 290°) \cdot (10, 48°)$

## Reasoning and communication

5 Change each of the following to polar form and find their dot product, correct to 2 decimal places if necessary.

- a $(1, 4)$ and $(4, 3)$
- b $(6, 2)$ and $(8, 4)$
- c $(-5, 3)$ and $(2, 6)$
- d $(4, -3)$ and $(6, 2)$
- e $(-2, -7)$ and $(-3, 5)$
- f $(-4, 6)$ and $(3, -8)$

# 4.02 THE COMPONENT FORM

As you have seen, the component form of vectors is easier to use for addition or subtraction of vectors. This is also true for the dot product.

**IMPORTANT**

**Scalar product of vectors in component form**

For two vectors $\mathbf{v}_1 = (a_1, a_2)$ and $\mathbf{v}_2 = (b_1, b_2)$, the **scalar product** or **dot product** is given by:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (a_1, a_2) \cdot (b_1, b_2) = a_1 \times b_1 + a_2 \times b_2$$

Use of either form of the dot product will give the same answer.

## Example 4

If $\mathbf{a} = (1, -1)$ and $\mathbf{b} = (3, 4)$ calculate $\mathbf{a} \cdot \mathbf{b}$ using:

**a** the component form **b** the polar form.

**c** What do you find?

### Solution

**a** Write the rule. $(a_1, a_2) \cdot (b_1, b_2) = a_1 \times b_1 + a_2 \times b_2$

Substitute in the values. $(1, -1) \cdot (3, 4) = 1 \times 3 + (-1) \times 4$

Calculate the answer. $= -1$

**b** Change $(1, -1)$ to polar form.

$$\tan(\theta) = \frac{-1}{1} = -1$$

Find $\theta$. $\theta = -45°$ or $315°$

Find the magnitude.

$$r = \sqrt{1^2 + (-1)^2} = \sqrt{2} = 1.41\ldots$$

Change $(3, 4)$ to polar form. $\tan(\theta) = \frac{4}{3} = 1\frac{1}{3} = 1.33\ldots$

Find $\theta$. $\theta = 53.13\ldots°$

Find the magnitude.

$$r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Write the rule for the polar form. $(1.41\ldots, -45°) \cdot (5, 53.13\ldots°)$

The angle between the vectors is $98.13\ldots°$. $= 1.41\ldots \times 5 \times \cos(\theta)$

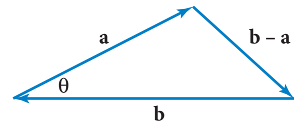
Find the answer, using the extended values in your calculator. $= 1.41\ldots \times 5 \times \cos(98.13\ldots°)$ $= -1$

**c** Write your findings. The answers are the same.

You can use the cosine rule [$c^2 = a^2 + b^2 - 2ab\cos(C)$] to prove that the polar form and the component form of the dot product are *always* the same.

Consider two vectors $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$ and their vector difference $\mathbf{b} - \mathbf{a}$.

This is shown by the triangle of vectors to the right.




In this case, $a = |\mathbf{a}|$, $b = |\mathbf{b}|$, $c = |\mathbf{b} - \mathbf{a}|$ and $\theta$ is the angle between **a** and **b**.

Now $\mathbf{b} - \mathbf{a} = (b_1 - a_1, b_2 - a_2)$, so using $r = \sqrt{a_1^2 + a_2^2}$ gives

$$a^2 = a_1^2 + a_2^2,\ b^2 = b_1^2 + b_2^2 \text{ and } c^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2.$$

Substitute all these into the cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos(C).$$

This gives

$$(a_1 - b_1)^2 + (a_2 - b_2)^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 - 2|\mathbf{a}| \times |\mathbf{b}|\cos(\theta)$$

Multiply out the brackets to get the following.

$$a_1^2 - 2a_1b_1 + b_1^2 + a_2^2 - 2a_2b_2 + b_2^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 - 2|\mathbf{a}| \times |\mathbf{b}|\cos(\theta)$$

So $$-2a_1b_1 - 2a_2b_2 = -2|\mathbf{a}| \times |\mathbf{b}|\cos(\theta)$$

Divide by –2 to get $$a_1b_1 + a_2b_2 = |\mathbf{a}| \times |\mathbf{b}|\cos(\theta)$$ **QED**

## Example 5

Calculate $\mathbf{q} \cdot \mathbf{r}$ when:

a $\mathbf{q} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{r} = \mathbf{i} + 6\mathbf{j}$ b $\mathbf{q} = (2, 4)$ and $\mathbf{r} = (2, 30°)$.

### Solution

a Write **q** and **r** in component form. $\mathbf{q} = (2, -3)$, $\mathbf{r} = (1, 6)$

Apply the rule in component form. $\mathbf{q} \cdot \mathbf{r} = 2 \times 1 + (-3) \times 6$

Evaluate. $= -16$

b Convert **r** from polar to component form. $\mathbf{r} = (2\cos(30°), 2\sin(30°))$ $= (\sqrt{3}, 1)$

Apply the rule in component form. $\mathbf{q} \cdot \mathbf{r} = 2 \times \sqrt{3} + 4 \times 1$

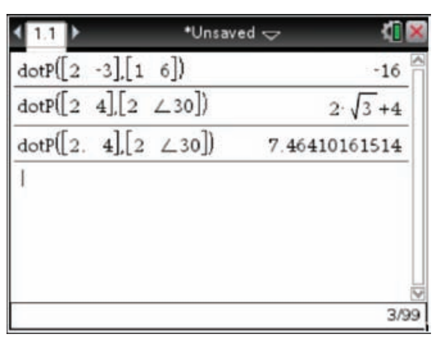
Simplify. $= 4 + 2\sqrt{3}$

**TI-Nspire CAS**

Use a Calculator page.

You can do both parts without changing the form of the vectors.

If your calculation mode is set on auto, put in a decimal point to get the approximate answer.



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**ClassPad**

Use the $\overset{\text{Main}}{\sqrt{\alpha}}$ application.

First set the calculator to degrees (**Deg**) and **Standard**.

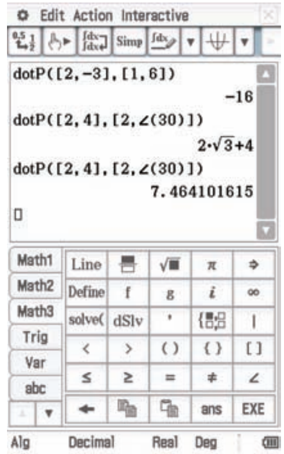
Tap **Action**, then **Vector** and then **dotP**.

Enter the two vectors, separated by a comma.

Tap EXE or press the EXE key.

Remember to put each vector in square brackets.

The third result was obtained by setting the calculator to **Decimal**.




The formula for scalar products provides a convenient way of calculating the angle between two vectors.

State the definition. $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \cos(\theta)$

Rearrange. $\cos(\theta) = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

**IMPORTANT**

**Angle between two vectors**

For vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$, the angle $\theta$ between **a** and **b** is given by

$$\cos(\theta) = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \text{ or } \cos(\theta) = \frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2 + a_2^2} \times \sqrt{b_1^2 + b_2^2}}$$

## Example 6

Calculate the angle between $\mathbf{g} = (2, 6)$ and $\mathbf{h} = (7, 1)$.

### Solution

Calculate the magnitude of the vectors.

$$|\mathbf{g}| = \sqrt{2^2 + 6^2} = 2\sqrt{10}$$

$$|\mathbf{h}| = \sqrt{7^2 + 1^2} = 5\sqrt{2}$$

Calculate the dot product.

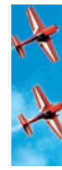
$$\mathbf{g}\cdot\mathbf{h} = 2 \times 7 + 6 \times 1 = 20$$

Write the formula.

$$\cos(\theta) = \frac{\mathbf{g}\cdot\mathbf{h}}{|\mathbf{g}||\mathbf{h}|}$$

Substitute in the required values.

$$= \frac{20}{2\sqrt{10} \times 5\sqrt{2}}$$



Evaluate.

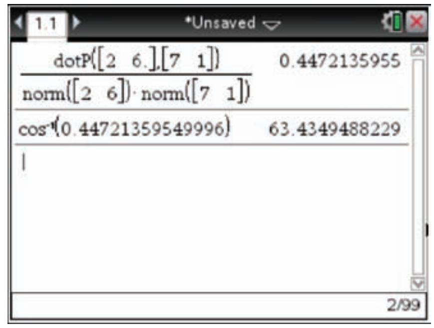
$$= \frac{20}{20\sqrt{5}}$$
$$= \frac{1}{\sqrt{5}}$$

Use $\cos^{-1}$ to calculate $\theta$.

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$
$$\approx 63.43°$$

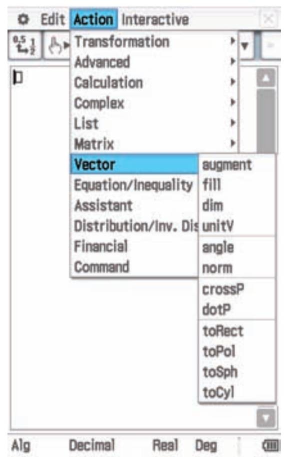
## TI-Nspire CAS

Use a Calculator page.
Use the fraction template or brackets and division.
You can use the norm() function to find magnitudes. You can type it in or use menu, 7: Matrix & Vector, 7: Norms and 1: Norm.
Remember to press TRIG to get $\cos^{-1}$.
Press ctrl (–) to get the previous answer in the last part.

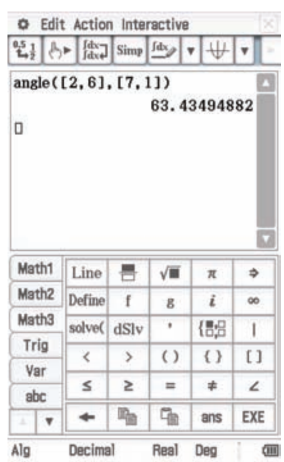



## ClassPad

Use the Main $\sqrt{\alpha}$ application.
First set the calculator to degrees (**Deg**) and **Standard**.
Tap **Action**, then **Vector** and then **angle**.




Enter the two vectors, separated by a comma.
Tap EXE or press the EXE key.





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# EXERCISE 4.02 The component form

## Concepts and techniques

Component form of the dot product

1 Example 4 In each of the following cases, calculate $\mathbf{a} \cdot \mathbf{b}$ using both the component and polar forms.

a $\mathbf{a} = (2, 0)$ and $\mathbf{b} = (5, 12)$
b $\mathbf{a} = (3, 0)$ and $\mathbf{b} = (4, 3)$
c $\mathbf{a} = (-1, 0)$ and $\mathbf{b} = (-5, 7)$
d $\mathbf{a} = (0, 1)$ and $\mathbf{b} = (8, 2)$
e $\mathbf{a} = (0, -3)$ and $\mathbf{b} = (5, -6)$

2 Find the following scalar products.

a $(9, 120°) \cdot (16, 48°)$
b $(3, 185°) \cdot (16, 200°)$
c $(19, 300°) \cdot (6, 240°)$

3 Example 5 Calculate the scalar product of each of the following pairs of vectors.

a $5\mathbf{i} + \mathbf{j}$ and $3\mathbf{i} + 4\mathbf{j}$
b $2\mathbf{i} + 7\mathbf{j}$ and $8\mathbf{i} + \mathbf{j}$
c $6\mathbf{i} + 2\mathbf{j}$ and $3\mathbf{i} + 5\mathbf{j}$
d $-3\mathbf{i} + 3\mathbf{j}$ and $4\mathbf{i} + 7\mathbf{j}$
e $4\mathbf{i} - 2\mathbf{j}$ and $3\mathbf{i} - 3\mathbf{j}$
f $-9\mathbf{i} + 2\mathbf{j}$ and $-3\mathbf{i} - 11\mathbf{j}$

4 Calculate the dot products of each of the following pairs of vectors.

a $(1, 5)$ and $(5, 0)$
b $(3, 7)$ and $(0, 2)$
c $(-2, 6)$ and $(4, 2)$
d $(7, -3)$ and $(5, 5)$
e $(6, -2)$ and $(3, 8)$
f $(6, -1)$ and $(-3, -7)$

5 CAS Find the following scalar products.

a $\begin{bmatrix} 3 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -4 \end{bmatrix}$
b $\begin{bmatrix} -2 \\ 10 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix}$
c $\begin{bmatrix} -6 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ -6 \end{bmatrix}$
d $\begin{bmatrix} 3 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 3 \end{bmatrix}$
e $\begin{bmatrix} 2 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -9 \end{bmatrix}$
f $\begin{bmatrix} -3 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 8 \end{bmatrix}$

6 Calculate the scalar product of the following pairs of vectors, correct to two decimal places if necessary.

a $(4, 3)$ and $(6, 60°)$
b $(-1, 5)$ and $(4, 45°)$
c $\left(7, \frac{\pi}{6}\right)$ and $(3, -2)$
d $(5, 5)$ and $\left(3, \frac{7\pi}{8}\right)$
e $(6, 115°)$ and $(-4, 4)$
f $(5, -2)$ and $(3, 98°)$

7 Example 6 Calculate the angles between the following pairs of vectors, correct to two decimal places if necessary.

a $(3, 4)$ and $(5, 12)$
b $(1, 3)$ and $(3, 6)$
c $(5, -4)$ and $(-2, 1)$
d $(1, 5)$ and $(4, -6)$
e $(-3, 5)$ and $(4, -2)$
f $(-2, 5)$ and $(3, -2)$

8 CAS Calculate the angles between the following pairs of vectors, correct to two decimal places if necessary.

a $(2, 6)$ and $(-3, 5)$
b $(-4, 5)$ and $(5, 3)$
c $(2, -8)$ and $(-3, 7)$
d $(4, 9)$ and $(3, -2)$
e $(-5, 3)$ and $(2, -4)$
f $(-6, -8)$ and $(-5, -2)$

## Reasoning and communication

9 Use the scalar product to show that $(-4, 0)$ and $(0, 5)$ are perpendicular.

10 Consider the scalar product of unit vectors with directions $A$ and $B$. In polar form, the vectors are $(1, A)$ and $(1, B)$. Use the scalar product in component form to find an expression for the cosine of the angle between the vectors. Use $\cos(A - B)$.

# 4.03 PROPERTIES OF THE SCALAR PRODUCT

The scalar product of two vectors has a number of very important properties. You may have noticed some of these in the previous sections.

**IMPORTANT**

**Properties of the scalar product**

1 The dot product may be calculated from either polar or component forms of vectors as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \cos(\theta)$$

where $\theta$ is the angle between **a** and **b** or

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$$

where $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$.

2 The scalar product is a real number, *not a vector*.

3 The scalar product is commutative.

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

4 The scalar product is distributive over vector addition.

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

5 $m(\mathbf{a} \cdot \mathbf{b}) = (m\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (m\mathbf{b})$ for all real $m$ and vectors **a** and **b**.

6 $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$

You can verify properties of the scalar product for particular vectors.

## Example 7

Show that the scalar product is commutative for the vectors $\mathbf{p} = (2, 7)$ and $\mathbf{q} = (3, -8)$.

### Solution

| | |
|---|---|
| Calculate $\mathbf{p} \cdot \mathbf{q}$. | $\mathbf{p} \cdot \mathbf{q} = 2 \times 3 + 7 \times (-8) = -50$ |
| Calculate $\mathbf{q} \cdot \mathbf{p}$. | $\mathbf{q} \cdot \mathbf{p} = 3 \times 2 + (-8) \times 7 = -50$ |
| State the result. | $\mathbf{p} \cdot \mathbf{q} = \mathbf{q} \cdot \mathbf{p}$ |

To prove the properties of scalar products you need to show that they are true of all vectors.

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## Example 8

Prove that the scalar product is distributive over vector addition.

### Solution

| | |
|---|---|
| Write down what is required to be proved. | **RTP**<br>$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ |
| | **Proof** |
| Identify the vectors. | Let $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$ and $\mathbf{c} = (c_1, c_2)$ |
| Write the LHS of the equation in component form. | $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = (a_1, a_2) \cdot (b_1 + c_1, b_2 + c_2)$ |
| Apply the rule for dot products. | $= a_1(b_1 + c_1) + a_2(b_2 + c_2)$<br>$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2$ |
| Write the RHS of the equation in component form. | $\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = (a_1, a_2) \cdot (b_1, b_2) + (a_1, a_2) \cdot (c_1, c_2)$ |
| Apply the rule for dot products. | $= a_1b_1 + a_2b_2 + a_1c_1 + a_2c_2$<br>$= \mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$ |
| | **QED** |

# EXERCISE 4.03 Properties of the scalar product

## Concepts and techniques

WS Properties of the dot product

1 Example 7 Use the following pairs of vectors to demonstrate that the scalar product is commutative (i.e. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$).

a $(2, 5)$ and $(1, 4)$
b $(-2, 3)$ and $(4, 5)$
c $(6, 7)$ and $(-3, -4)$
d $(-3, 4)$ and $(9, -8)$

2 Use the following sets of vectors to show that the scalar product is distributive over vector addition (i.e. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$).

a $(3, 7)$, $(2, 5)$ and $(4, 1)$
b $(2, 3.5)$, $(4.1, 2)$ and $(1.2, 3.4)$
c $(6.1, -1.8)$, $(5, 2)$ and $(-2, -3)$
d $(4.6, 9.2)$, $(-5.5, -4.6)$ and $(2, 0)$

3 Use the following pairs of vectors to demonstrate that $m(\mathbf{a} \cdot \mathbf{b}) = (m\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (m\mathbf{b})$, where $m$ is a real number and $\mathbf{a}$ and $\mathbf{b}$ are vectors.

a $(2, 5)$, $(3, 4)$ and $m = 3$
b $(1, 6)$, $(8, 3)$ and $m = 5$
c $(4, 1)$, $(0, -6)$ and $m = -2$
d $(-5, 9)$, $(7, 2)$ and $m = 2.5$

4 Use the following vectors to show that $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$.

a $(1, 2)$
b $(-3, 4)$
c $(4, -7)$
d $(-3, -6)$

## Reasoning and communication

5 Example 8 Prove that $\mathbf{q} \cdot \mathbf{r} = \mathbf{r} \cdot \mathbf{q}$, where $\mathbf{q} = (q_1, q_2)$ and $\mathbf{r} = (r_1, r_2)$.

6 Prove that $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, where $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$.

7 Prove that $m(\mathbf{a} \cdot \mathbf{b}) = (m\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (m\mathbf{b})$ for all real $m$ and vectors $\mathbf{a}$ and $\mathbf{b}$.

8 Prove that $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$.

9 Use the polar form of the dot product to show that $\mathbf{a} \cdot (-\mathbf{b}) = -(\mathbf{a} \cdot \mathbf{b})$

# 4.04 PARALLEL AND PERPENDICULAR VECTORS

## INVESTIGATION Parallel and perpendicular vectors

1 Make up some vectors that are parallel.

2 Find the scalar products of parallel vectors.

3 What do you find?

4 Choose some vectors that are perpendicular.

5 Find the scalar products of perpendicular vectors.

6 What do you find?

Consider the scalar product of two vectors **a** and **b**.

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

If $\mathbf{a} \cdot \mathbf{b} = 0$, then either $|\mathbf{a}| = 0$ or $|\mathbf{b}| = 0$ or $\cos(\theta) = 0$.

So, if **a** and **b** are nonzero, $\theta = 90°$ because $\cos(90°) = 0$.

### Example 9

Show that the vectors (2, 7) and (−7, 2) are perpendicular.

#### Solution

| | |
|---|---|
| Calculate the dot product. | $(2, 7) \cdot (-7, 2) = 2 \times (-7) + (-7) \times 2$<br>$= 0$ |
| State the result. | (2, 7) and (−7, 2) are both nonzero, so $\cos(\theta) = 0$.<br>Therefore they are perpendicular. |


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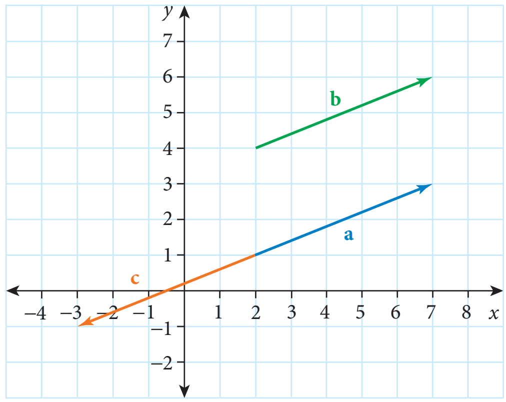
The diagram on the right shows three vectors: **a**, **b** and **c**.




The three vectors are parallel. Vectors **a** and **b** are in the same direction so the angle between them is 0°. Vectors **a** and **c** are in the opposite direction, so the angle between them is 180°.

Since $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$:

- if $\theta = 0°$, then $\cos(\theta) = 1$, so $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$
- if $\theta = 180°$, then $\cos(\theta) = -1$, so $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}||\mathbf{b}|$

**IMPORTANT**

**Perpendicular and parallel vectors**

If **a** and **b** are perpendicular (or **orthogonal**), then $\mathbf{a} \cdot \mathbf{b} = 0$

If **a** and **b** are parallel, then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$ or $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}||\mathbf{b}|$

For the unit vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$:

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$$

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$$

The dot products of the unit vectors **i** and **j** are as shown because any vector is parallel to itself and **i** and **j** are perpendicular. Sometimes vectors in the opposite direction are called antiparallel. This is the case where $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}||\mathbf{b}|$.

## Example 10

Determine if the vectors $\mathbf{c} = (5, -4)$ and $\mathbf{d} = (-5, 4)$ are perpendicular, parallel or neither.

### Solution

| | |
|---|---|
| Calculate the magnitude of each vector. | $\lvert\mathbf{c}\rvert = \sqrt{5^2 + (-4)^2} = \sqrt{41}$ <br> $\lvert\mathbf{d}\rvert = \sqrt{(-5)^2 + 4^2} = \sqrt{41}$ |
| Calculate the dot product. | $\mathbf{c} \cdot \mathbf{d} = (5, -4) \cdot (-5, 4)$ <br> $= 5 \times (-5) + (-4) \times 4$ <br> $= -41$ |
| Calculate $\lvert\mathbf{c}\rvert\lvert\mathbf{d}\rvert$. | $\lvert\mathbf{c}\rvert\lvert\mathbf{d}\rvert = \sqrt{41} \times \sqrt{41} = 41$ |
| Compare $\lvert\mathbf{c}\rvert\lvert\mathbf{d}\rvert$ with the dot product. | $\lvert\mathbf{c}\rvert\lvert\mathbf{d}\rvert = -\mathbf{c} \cdot \mathbf{d}$ |
| State the result. | **c** and **d** are parallel but in opposite directions (antiparallel). |

# EXERCISE 4.04 Parallel and perpendicular vectors

WS Parallel and perpendicular vectors

## Concepts and techniques

1 The vector $\mathbf{a} = 3\mathbf{i}$. Which of the following are orthogonal to $\mathbf{a}$ (at right angles)?

A $-3\mathbf{i}$ B $-3\mathbf{j}$ C $3\mathbf{i} + 3\mathbf{j}$ D $-3\mathbf{i} - 3\mathbf{j}$ E 0

2 Example 9 Show that the following pairs of vectors are perpendicular.

a $(2, 0)$ and $(0, 6)$
b $(4, -2)$ and $(2, 4)$
c $(-5, 5)$ and $(5, 5)$
d $(-12, 0)$ and $(0, 16)$
e $(3\sqrt{5}, 2\sqrt{3})$ and $(-2\sqrt{3}, 3\sqrt{5})$
f $\left(\frac{2}{3}, \frac{-4}{5}\right)$ and $\left(\frac{4}{5}, \frac{2}{3}\right)$

3 Show that the following pairs of vectors are perpendicular.

a $\mathbf{i} - 3\mathbf{j}$ and $3\mathbf{i} + \mathbf{j}$
b $7\mathbf{i}$ and $4\mathbf{j}$
c $9\mathbf{i} + 6\mathbf{j}$ and $6\mathbf{i} - 9\mathbf{j}$
d $\begin{bmatrix} -2 \\ 8 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$
e $\begin{bmatrix} \frac{1}{4} \\ \frac{3}{8} \end{bmatrix}$ and $\begin{bmatrix} -8 \\ \frac{16}{3} \end{bmatrix}$
f $\begin{bmatrix} 5\sqrt{6} \\ -3\sqrt{10} \end{bmatrix}$ and $\begin{bmatrix} 3\sqrt{3} \\ 3\sqrt{5} \end{bmatrix}$

4 Use the scalar product to show that the following pairs of vectors are parallel or antiparallel.

a $(3, 5)$ and $(6, 10)$
b $(-4, 2)$ and $(8, -4)$
c $(7, 3)$ and $(9\frac{1}{3}, 4)$
d $(-5, 2)$ and $(7, -2.8)$
e $(4\sqrt{6}, 2\sqrt{3})$ and $(-12\sqrt{2}, -6)$
f $\left(\frac{2}{3}, \frac{-4}{5}\right)$ and $\left(\frac{4}{5}, -\frac{24}{25}\right)$

5 Use the scalar product to show that the following pairs of vectors are parallel or antiparallel.

a $2\mathbf{i} + 6\mathbf{j}$ and $\mathbf{i} + 3\mathbf{j}$
b $4\mathbf{i} - 5\mathbf{j}$ and $-4.8\mathbf{i} + 6\mathbf{j}$
c $4\mathbf{i} + 9\mathbf{j}$ and $2\frac{2}{3}\mathbf{i} + 6\mathbf{j}$
d $\begin{bmatrix} 5 \\ -7 \end{bmatrix}$ and $\begin{bmatrix} 10 \\ 14 \end{bmatrix}$
e $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ 2\frac{2}{3} \end{bmatrix}$
f $\begin{bmatrix} 3\sqrt{6} \\ 4\sqrt{15} \end{bmatrix}$ and $\begin{bmatrix} 18\sqrt{2} \\ 24\sqrt{5} \end{bmatrix}$

6 Example 10 Determine if the following pairs of vectors are perpendicular, parallel or neither.

a $(5, -4)$ and $(-4, 5)$
b $(8, -3)$ and $(-12, 4.5)$
c $(4, -3)$ and $(-6, 5)$
d $(-6, 4)$ and $(-15, 10)$
e $\left(\frac{3}{4}, \frac{5}{8}\right)$ and $(-5, 6)$
f $\left(3\sqrt{10}, 2\sqrt{3}\right)$ and $\left(4\sqrt{2}, 5\sqrt{6}\right)$

## Reasoning and communication

7 Vectors $\mathbf{p} = (a, -3)$ and $\mathbf{q} = (8, 6)$ are perpendicular. Calculate the possible values of $a$.

8 Vectors $\mathbf{r} = (4, b)$ and $\mathbf{t} = (3, 6)$ are parallel. Calculate the possible values of $b$.

9 The vector $\mathbf{m}$ is fixed and its magnitude is 4 while the vector $\mathbf{n}$ is free to rotate and has a magnitude of 5. What are the maximum and minimum values of the dot product $\mathbf{m} \cdot \mathbf{n}$ as $\mathbf{n}$ rotates through all possible positions? What positions of $\mathbf{m}$ and $\mathbf{n}$ lead to these values?

10 Consider the points $A(-1, 6)$, $B(-3, -2)$ and $C(7, 3)$. Calculate the angle between $\mathbf{BA}$ and $\mathbf{BC}$.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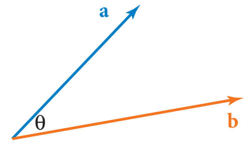
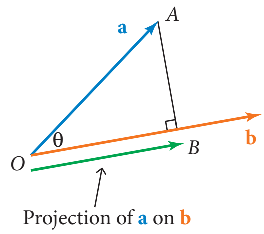
# 4.05 PROJECTION OF A VECTOR




Consider the two vectors **a** and **b** shown here. The angle between the vectors is θ.

If a line is drawn from **a** perpendicular to **b** as shown by $AB$ here, the part of **b** indicated by the green arrow ($OB$) is called the **projection** of **a** on **b**. It is also called the projection of **a** in the direction of **b**.

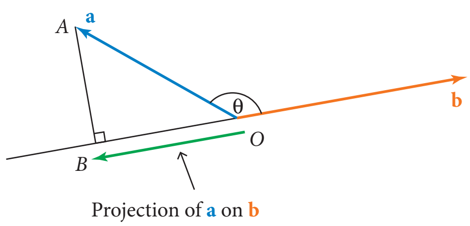



If the angle between the two vectors is obtuse, then the projection is in the opposite direction to vector **b** and will be negative.

From the right-angled triangle $OAB$ and using trigonometry:

$$\cos(\theta) = \frac{OB}{|\mathbf{a}|}$$

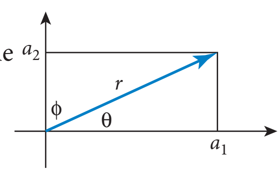
or $$OB = |\mathbf{a}|\cos(\theta)$$




So, if the value of θ is known, the projection $p$ of **a** on **b** is given by:

$$p = |\mathbf{a}|\cos(\theta)$$

Since a projection is in a particular direction, it is actually a vector, but the magnitude is generally referred to as the projection. The phrase **vector projection** may be used to mean the vector with magnitude $|\mathbf{a}|\cos(\theta)$ in the direction of **b**, namely, $|\mathbf{a}|\cos(\theta)\hat{\mathbf{b}}$.




As can be seen from the diagram on the right, the component form of a vector is actually the projection onto the $x$-axis and the projection onto the $y$-axis.

$\mathbf{a} = \begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, where $a_1 = r\cos(\theta)$ and $a_2 = r\sin(\theta)$

But $\sin(\theta) = \cos(90° - \theta) = \cos(\phi)$ and $r = |\mathbf{a}|$, so

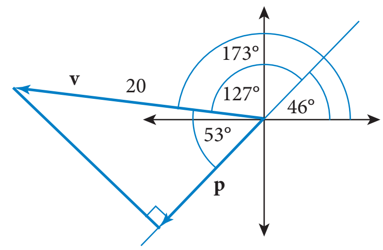
$a_1 = |\mathbf{a}|\cos(\theta)$ and $a_2 = |\mathbf{a}|\cos(\phi)$, the projections on the $x$ and $y$ axes (or on **i** and **j**).

## Example 11

Find the projection in the direction 46° of a vector of magnitude 20 in the direction 173°.

### Solution

Sketch a diagram showing the vector in the direction 173° and the direction of the projection at 46°.
Indicate the projection $p$ and the vector $\mathbf{v}$ on the diagram.



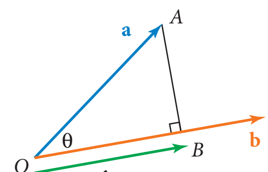

Calculate the projection. $p = 20\cos(127°)$

Evaluate. $\approx -12.0$

The negative sign indicates that the projection is in the opposite direction. It could also have been obtained using $-20\cos(53°)$.

State the result. The required projection is −12.0.

Consider again the diagram used before Example **11**.




The projection of $\mathbf{a}$ on $\mathbf{b}$ is given by $OB = |\mathbf{a}|\cos(\theta)$

But $\cos(\theta) = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

So $OB = |\mathbf{a}|\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$

You can write this as $OB = \mathbf{a}\cdot\dfrac{\mathbf{b}}{|\mathbf{b}|}$

But $\hat{\mathbf{b}} = \dfrac{1}{|\mathbf{b}|}\mathbf{b}$

So $OB = \mathbf{a}\cdot\hat{\mathbf{b}}$

**IMPORTANT**

The magnitude of a **projection**, **p**, of **a** on **b** is the scalar product of **a** and a unit vector in the direction of **b**.

$$p = \mathbf{a}\cdot\hat{\mathbf{b}}$$

where $\hat{\mathbf{b}}$ is a unit vector in the direction of **b**.

Alternatively: $p = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$

The **vector projection** is $\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\hat{\mathbf{b}} = |\mathbf{a}|\cos(\theta)\hat{\mathbf{b}}$, where $\theta$ is the angle between **a** and **b**.



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## Example 12

Find the projection of $\mathbf{m} = (5, 7)$ on $\mathbf{n} = (6, 2)$.

### Solution

Find a unit vector in the direction of $\mathbf{n}$.

$$\begin{aligned}\hat{\mathbf{n}} &= \frac{1}{|\mathbf{n}|}\mathbf{n}\\ &= \frac{1}{\sqrt{6^2+2^2}}(6, 2)\\ &= \frac{1}{2\sqrt{10}}(6, 2)\\ &= \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)\end{aligned}$$

Use the formula for the projection of $\mathbf{m}$ on $\mathbf{n}$.

$$p = \mathbf{m}\cdot\hat{\mathbf{n}}$$

Substitute in the known values.

$$\begin{aligned}&= (5, 7)\cdot\left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)\\ &= \frac{15}{\sqrt{10}} + \frac{7}{\sqrt{10}}\end{aligned}$$

Evaluate.

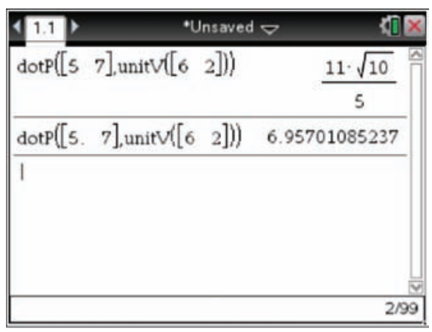
$$= \frac{22}{\sqrt{10}}$$

Rationalise the denominator.

$$\begin{aligned}&= \frac{11\sqrt{10}}{5}\\ &\approx 6.96\end{aligned}$$

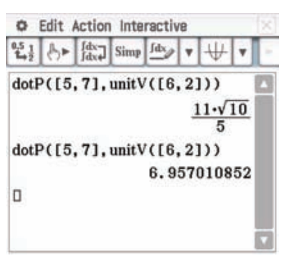
**TI-Nspire CAS**

You can find the projection using the dot product and unit vector functions.
You type unitV() or use menu, 7: Matrix & Vector, C: Vector and 1: Unit Vector.



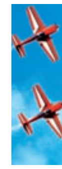

**ClassPad**

Use the Main $\sqrt{\alpha}$ application. Set the calculator to **Standard** for an exact result (which may be a surd or a fraction) and **Decimal** for an approximate answer. Mathematicians generally prefer exact answers.




Take the dot product of $\mathbf{m}$ and $\hat{\mathbf{n}}$.

Tap **Action** and then **Vector** to get **dotP** and **unitV**.



# EXERCISE 4.05 Projection of a vector

## Concepts and techniques

WS Projections

1 The projection of $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j}$ on the positive direction of the $x$-axis is:
A $-2$ B $0$ C $2$ D $4$ E $2\sqrt{5}$

2 Example 11 The projection of $\mathbf{m} = (4, 60°)$ on $\mathbf{n} = (11, 240°)$ is:
A $-11$ B $-4$ C $0$ D $4$ E $11$

3 The projection of $\mathbf{p} = (5, 120°)$ on $\mathbf{q} = (7, 30°)$ is:
A $-7$ B $-5$ C $0$ D $5$ E $7$

4 Example 12 The projection of $2\mathbf{i} + 4\mathbf{j}$ on $-\mathbf{i} + \frac{1}{2}\mathbf{j}$ is:
A $-1$ B $0$ C $\frac{1}{2}$ D $2$ E $4$

5 Find the projection in the direction 60° of a vector of magnitude 18 and direction 86°.

6 Find the projection in the direction $\frac{2\pi}{3}$ of a vector of magnitude 16 in the direction 82°.

7 Find the projection in the direction 206° of the vector (8, 100°).

8 Find the projection of (32, 178°) in the direction 200°.

9 What is the projection of (16, 35°) in the direction 305°?

10 What is the projection of (19, 116°) in the direction 116°?

11 Find the projection of $\mathbf{a} = (1, 5)$ on $\mathbf{b} = (3, 4)$.

12 Find the projection of $-10\mathbf{i} - 2\mathbf{j}$ on $5\mathbf{i} + 12\mathbf{j}$.

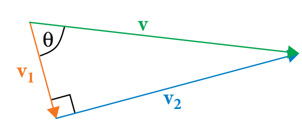
13 Find the projection of $\mathbf{m} = (-3, 5)$ on $\mathbf{n} = (-7, 2)$.

14 Find the projection of $\mathbf{u} = (4, -3)$ on $\mathbf{v} = (-6, -2)$.

15 Find the projection of $-9\mathbf{i} - 3\mathbf{j}$ on $-\mathbf{i} - 2\mathbf{j}$.

# 4.06 RESOLUTION OF FORCES

Practical problems with vectors often involve combining several vectors. As you have seen it is easier to combine vectors in component form than in polar form. The natural direction to use may not be in the directions of $\mathbf{i}$ or $\mathbf{j}$. In this case we change the vectors to a sum of projections that are perpendicular to each other.




In the diagram above right, the vector $\mathbf{v}$ is the sum of $\mathbf{v}_1$ and $\mathbf{v}_2$. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are perpendicular to each other. If the angle between $\mathbf{v}$ and $\mathbf{v}_1$ is θ, then $v_1 = v\cos(\theta)$ and $v_2 = v\sin(\theta)$.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## IMPORTANT

**Orthogonal components** of a vector are projections of the vector in perpendicular directions. Writing a given vector as a sum of orthogonal components is called **resolving** the vector into orthogonal components.

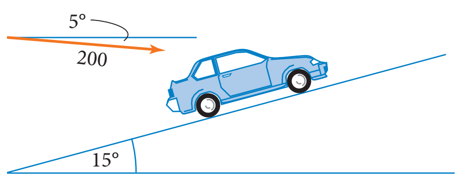
### Example 13

Some people are pushing a car up a hill with a slope of 15°. One person who is pushing is applying a force of 200 N at an angle 5° downwards from the horizontal. Resolve this force into components parallel and perpendicular to the road.

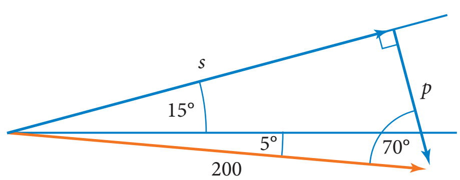
Alamy/Juice Images

### Solution

Sketch a diagram to represent the given information.




Draw another diagram to show the orthogonal components. From the diagram, you can see that the angles between the force of 200 N and its orthogonal components $s$ and $p$ are 20° and 70° respectively.



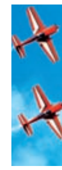

Calculate the projection $s$. $s = 200 \cos (20°)$

Evaluate. $\approx 187.9$

Calculate the other component, $p$. $p = 200 \sin (20°)$

Evaluate. $\approx 68.4$

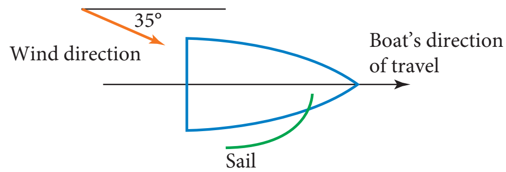
State the result. The components of the 200 N force parallel and perpendicular to the road are about 187.9 N and 68.4 N respectively.



In some cases, you will not know the actual force involved, but will want to find part of a force in a particular direction.

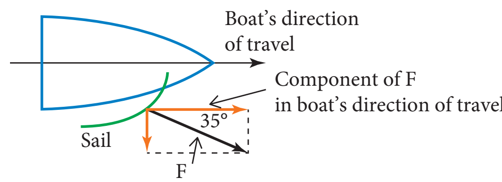
## Example 14

A boat is sailing in a particular direction with the wind direction 35° to the south of the direction of travel, as shown in the diagram below. The wind exerts a force **F** on the boat's sail. Calculate the percentage of the force of the wind that is in the direction of travel of the boat.




### Solution

Redraw the diagram showing **F** and its components.




Find the forward component.

$F_d = F\cos(35°)$
$= 0.8191\ldots F$

State the result.

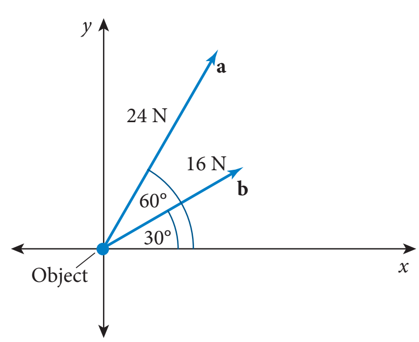
The component of the force of the wind in the direction of travel of the boat is about 82% of the force exerted by the wind.

## Example 15

One person applies a force of 24 newtons to an object at an angle of 60° upwards from the horizontal. Another person applies a force of 16 N in the same plane at an angle of 30° upwards from the horizontal. What is the resultant force acting on the object?

### Solution

Sketch a diagram, making $x$ the horizontal direction and $y$ the vertical direction.
Let the 24 N force be **a**. Show **a** acting at 60° from the horizontal.
Show the force of 16 N as **b**, acting at 30° from the horizontal.
In order to add the forces, each can be resolved into its $x$ and $y$ components using $x = r\cos(\theta)$ and $y = r\sin(\theta)$.





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Resolve **a**.

$\mathbf{a} = (24\cos(60°), 24\sin(60°))$

Evaluate using $\cos(60°) = \frac{1}{2}$ and $\sin(60°) = \frac{\sqrt{3}}{2}$.

$= (12, 12\sqrt{3})$

Resolve **b**.

$\mathbf{b} = (16\cos(30°), 16\sin(30°))$

Evaluate using $\cos(30°) = \frac{\sqrt{3}}{2}$ and $\sin(30°) = \frac{1}{2}$.

$= (8\sqrt{3}, 8)$

Find the resultant, **r**.

$\mathbf{r} = (12, 12\sqrt{3}) + (8\sqrt{3}, 8)$

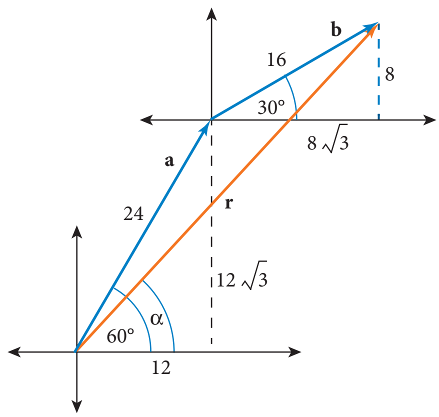
Add the $x$ and $y$ components separately.

$= (12 + 8\sqrt{3}, 12\sqrt{3} + 8)$

Evaluate and store the answers.

$= (25.86\ldots, 28.78\ldots)$

Use a diagram to show the addition of the vectors. Find the magnitude and direction of **r**.




Find $r = |\mathbf{r}|$.
Substitute the stored answers.

$$r^2 = x^2 + y^2$$
$$= 25.86\ldots^2 + 28.78\ldots^2$$
$$r = 38.69\ldots$$

Find the direction of **r**.

$$\tan(\alpha) = \frac{28.78\cdots}{25.86\cdots}$$

Use $\tan^{-1}$ to find $\alpha$.

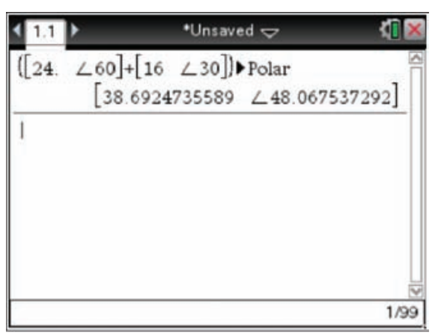
$$\alpha = 48.06\ldots°$$

Round and state the result.

The resultant force is approximately 38.7 N at 48.1° to the horizontal.

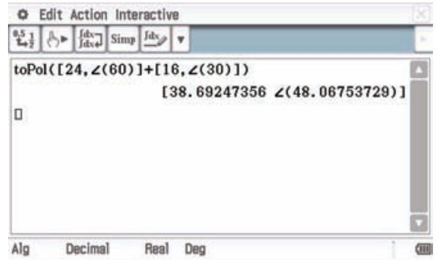
### TI-Nspire CAS

Use a Calculator page.
Write the addition of the vectors and express the answer in polar form.
Use a decimal point to force approximate calculation.
If you type the vectors make sure you use square brackets.



**ClassPad**

Use the Main $\sqrt{\alpha}$ application and add the vectors.
The answer must be in polar form, so start with **toPol**.
Then enter the vector sum,
toPol([24,∠(60)]+[16,∠(30)])
Tap **Action**, then **Vector** for **toPol**.




# EXERCISE 4.06 Resolution of forces

## Concepts and techniques

WS Resolution of forces

1 Example 14 An aeroplane is flying toward the north-east. Which of the following wind velocity vectors increases the plane's speed the most?

A $\mathbf{w}_1 = 4\mathbf{i} + \mathbf{j}$ B $\mathbf{w}_2 = -\mathbf{i} + 2\mathbf{j}$ C $\mathbf{w}_3 = \mathbf{i} - 8\mathbf{j}$
D $\mathbf{w}_4 = -10\mathbf{i} - 2\mathbf{j}$ E $\mathbf{w}_5 = -5\mathbf{i} - 2\mathbf{j}$

2 For the aeroplane described in question **1**, which of the following wind velocity vectors slows down the plane the most?

A $\mathbf{w}_1 = 4\mathbf{i} + \mathbf{j}$ B $\mathbf{w}_2 = -\mathbf{i} + 2\mathbf{j}$ C $\mathbf{w}_3 = \mathbf{i} - 8\mathbf{j}$
D $\mathbf{w}_4 = -10\mathbf{i} - 2\mathbf{j}$ E $\mathbf{w}_5 = -5\mathbf{i} - 2\mathbf{j}$

3 Example 13 Given vector $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$ and force vector $\mathbf{F}$ below, calculate:

i the component of $\mathbf{F}$ parallel to $\mathbf{v}$
ii the component of $\mathbf{F}$ perpendicular to $\mathbf{v}$.

a $\mathbf{F} = 4\mathbf{i} + \mathbf{j}$ b $\mathbf{F} = 0.5\mathbf{i} - 0.2\mathbf{j}$ c $\mathbf{F} = 16\mathbf{i} + 12\mathbf{j}$
d $\mathbf{F} = -0.4\mathbf{i} - 0.3\mathbf{j}$ e $\mathbf{F} = -5\mathbf{i} - 3\mathbf{j}$ f $\mathbf{F} = -8\mathbf{i} - 6\mathbf{j}$

4 The force on an object is give by $\mathbf{F} = -15\mathbf{j}$. For each vector $\mathbf{v}$ below, find:

i the component of $\mathbf{F}$ parallel to $\mathbf{v}$
ii the component of $\mathbf{F}$ perpendicular to $\mathbf{v}$

a $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$ b $\mathbf{v} = 5\mathbf{i} - \mathbf{j}$ c $\mathbf{v} = 6\mathbf{j}$ d $\mathbf{v} = 4\mathbf{i}$

## Reasoning and communication

5 Example 15 Three horizontal forces of 300 N, 400 N and 500 N are being exerted on an object. The 300 N and 500 N forces are exerted either side of the 400 N force at angles of 30° and 50° respectively. Find the total acting force and the direction of that force.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6 The winning team of a tug-of-war competition offers to take on two opponents. The teams working together have their ropes at an angle of 70° to each other and are exerting about the same force. The winning team is exerting a force of 1200 N but is easily overcome by the other two teams. The combined force of the other teams is actually 400 N more than the winning team. What forces were exerted by each of the two other teams?

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7 A 100 m sprint is run on an athletics track in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$. The wind velocity is $\mathbf{w} = 6\mathbf{i} + \mathbf{j}$ km/h. How much assistance (in km/h) do the athletes get from the wind in the direction of the race?

8 A car going up a hill inclined at 7° has a weight of 9000 N acting vertically downwards. Resolve the weight into components parallel and perpendicular to the slope.

9 In a physics experiment, a small shot put ball that is rolling down a plank sloped at 12° to the horizontal has a weight of 6 N. The total force acting on the ball is worked out from its acceleration to be 0.7 N acting down the slope. Resolve the weight into components parallel and perpendicular to the slope. The force acting down the slope is reduced by the force of friction acting back up the slope. What must the force of friction be?

10 A hang-glider is flying at a constant speed from a cliff-top down towards a long beach. The angle of flight is 5° down from the horizontal. The weight of the glider and its rider is 1100 N altogether. Find the drag acting on the hang-glider back along the flight path, assuming that the drag force of the air on the wing acts perpendicularly to the path.

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11 Before a boat on a trailer can be hitched to a car off the sand, is has to be pulled up the beach. The slope of the sand is 5° upwards and the person at the front is pulling upwards with a force of 400 N at an angle of 15° to the horizontal. Another person is pushing horizontally at the back with a force of 350 N. What is the total force moving the boat and trailer up the beach?

# 4.07 APPLICATIONS OF THE SCALAR PRODUCT

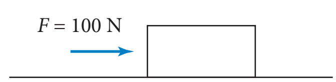
Projections and the dot product are used extensively in Physics to deal with vectors such as force, momentum, displacement and field strength.

The work done by a force is defined in physics as the product of the force and the displacement in the direction of the force. The displacement in the direction of the force is actually the projection of the displacement in the direction of the force. This means that $W = \mathbf{F} \cdot \mathbf{s}$, where $W$ is the work done, $\mathbf{F}$ is the force and $\mathbf{s}$ is the displacement.

The SI units of force, displacement and energy (work) are the newton (N), metre and joule (J) respectively.

## Example 16

Calculate the work done by a horizontal force of 100 N that moves an object 20 m horizontally.




### Solution

Use the formula for $W$ for displacement in the same direction as the force. $\quad W = F \times s$

Substitute in the given values. $\quad = 100 \text{ N} \times 20 \text{ m}$

Evaluate. $\quad = 2000 \text{ J}$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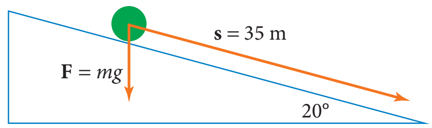
The energy gained by an object is the work done by the forces acting on the object.

## Example 17

How much energy is gained by a mass of 25 kg when it accelerates 35 m down a slope at 20° to the horizontal? (Note that the force exerted by gravity near the Earth is given by $\mathbf{F} = 9.81m$ newtons downwards, where $m$ is the mass of the object in g).

### Solution

Draw a diagram.




Find the vectors.

Gravitational force $= 25 \times 9.81$ N vertically
$= 245.25$ N vertically

Displacement $= 35$ m at 20° to the horizontal
$= 35$ m at 70° to the vertical

Find the dot product.

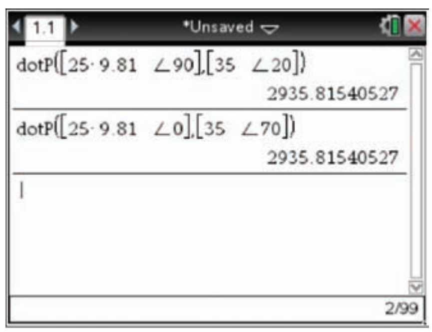
Work done $= \mathbf{F} \cdot \mathbf{s}$
$= 245.25 \times 35 \times \cos(70°)$
$= 2935.815\ldots$ J

Write the answer.

The mass gains about 2936 J of energy.

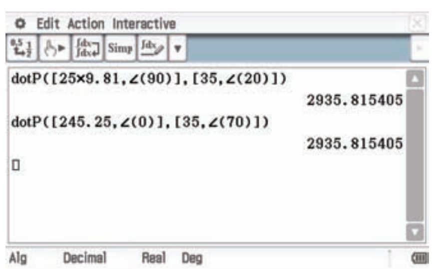
**TI-Nspire CAS**

Use a Calculator page.
Use the dot product, specifying the directions of the force and displacement relative to one or the other of the horizontal and vertical directions.




**ClassPad**

Use the Main $\sqrt{\alpha}$ application and **dotP**.
Set the calculator to **Decimal** and **Deg**.
You can specify the forces at angles of 90° and 20° or at angles of 0° and 70°.
You can specify the first force as either $25 \times 9.81$ or 245.25.
The screen on the right shows dotP([25×9.81, ∠(90)],[35, ∠(20)]) and dotP([245.25, ∠(0)],[35, ∠(70)])

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The scalar product can be used to prove geometric results.

### Example 18

Show that the diagonals of a rhombus intersect at right angles.

#### Solution

| | |
|---|---|
| State the equalities of a rhombus *ABCD*. | $\mathbf{AB} = \mathbf{DC}$, $\mathbf{BC} = \mathbf{AD}$ and $\lvert\mathbf{AB}\rvert^2 = \lvert\mathbf{DC}\rvert^2 = \lvert\mathbf{BC}\rvert^2 = \lvert\mathbf{AD}\rvert^2$ |
| Find the diagonal's dot product. | $\mathbf{AC} \cdot \mathbf{BD} = (\mathbf{AB} + \mathbf{BC}) \cdot (\mathbf{BA} + \mathbf{AD})$ |
| Expand. | $= \mathbf{AB} \cdot \mathbf{BA} + \mathbf{AB} \cdot \mathbf{AD} + \mathbf{BC} \cdot \mathbf{BA} + \mathbf{BC} \cdot \mathbf{AD}$ |
| Use equalities. | $= \mathbf{AB} \cdot \mathbf{BA} + \mathbf{AB} \cdot \mathbf{BC} + \mathbf{BC} \cdot \mathbf{BA} + \mathbf{BC} \cdot \mathbf{BC}$ |
| Use commutativity and simplify. | $= \mathbf{BC} \cdot (\mathbf{AB} + \mathbf{BA}) + \lvert\mathbf{BC}\rvert^2 - \lvert\mathbf{AB}\rvert^2$ |
| But $\mathbf{AB} + \mathbf{BA} = 0$ and $\lvert\mathbf{BC}\rvert^2 - \lvert\mathbf{AB}\rvert^2 = 0$. | $= 0$, so the diagonals are perpendicular. |

## EXERCISE 4.07 Applications of the scalar product

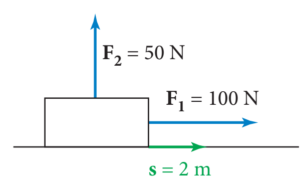
### Concepts and techniques

Applications of the dot product

You should use your CAS calculator wherever appropriate in this exercise.

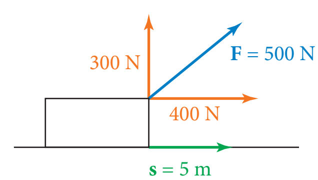
1 Example 16 An object is subjected to two forces as shown in the diagram on the right. If the object has a displacement of 2 m, then the work done by the forces is:

A 0 B 100 J C 200 J
D 224 J E 300 J

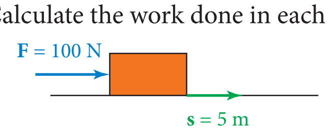


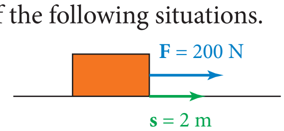

2 An object is subjected to a force of 500 N, as shown in the diagram on the right. The components of the force parallel and perpendicular to the displacement ($\mathbf{s}$) are 400 N and 300 N respectively. The work done by $\mathbf{F}$ is:

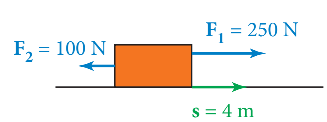
A 1500 J B 2000 J C 2500 J
D 3500 J E 4500 J

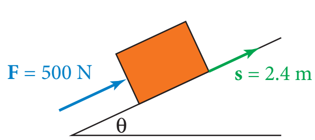


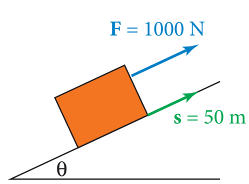

3 Calculate the work done in each of the following situations.

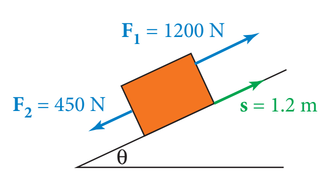
a



b



c



d



e



f


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4 Example 17 A ball is rolling down a slope at an angle of 34° to the horizontal. The weight of the ball is 40 N and this acts vertically downwards.

a What is the component of the weight that acts parallel to the direction of motion of the ball?

b How much work is done when the ball travels 1.8 m down the slope?

5 An object undergoes a displacement of $\mathbf{s} = 3\mathbf{i} + 4\mathbf{j}$ metres as a result of a force $\mathbf{F}$ newtons. Calculate the work done in each of the following cases.

a $\mathbf{F} = 4\mathbf{i} + \mathbf{j}$  b $\mathbf{F} = 2\mathbf{i} - 4\mathbf{j}$  c $\mathbf{F} = 9\mathbf{i} + 12\mathbf{j}$

d $\mathbf{F} = -3\mathbf{i} - 5\mathbf{j}$  e $\mathbf{F} = -4\mathbf{i} + 3\mathbf{j}$  f $\mathbf{F} = 4\mathbf{i} - 3\mathbf{j}$

## Reasoning and communication

6 A force of 500 N is exerted on an object. The force acts in a direction 45° above the horizontal and results in the object being displaced 20 m horizontally. Calculate the work done by the force.

7 A 36 kg object falls freely under the influence of gravity. Under gravity, $F = mg$, where $m$ is in kilograms and $g = 9.81\ \text{ms}^{-2}$. Calculate the work done when the object falls 28 m.

8 A force of 12 N causes an object to move 12 m at an angle of 60° to the direction of the force. What amount of energy is transferred?

9 Two forces of 30 N and 50 N act on an object. The 50 N force is at an angle of 35° to the 30 N force. The object moves 8 m at an angle of 15° to the 30 N force, away from the 50 N force.

a Find the total acting force.

b Find the energy transferred by the:

i total force  ii 30 N force  iii 50 N force.

c Compare the sizes of the answers in part **b** and interpret this information mathematically.

10 Three people are pushing parallel to the slope on the back of a car facing uphill at an angle of 5° to the horizontal. The weight of the car is about 8600 N.

a Resolve the weight into parallel and perpendicular components to the slope to find the average force they must each exert to overcome the weight pushing the car back down the hill.

b If the people stop pushing and let the car roll down the hill, calculate the work done when the car travels 120 m down the slope.

11 A boat and trailer are winched up an 8 m ramp inclined at 40° to the horizontal. The force of gravity on the boat and trailer is 1500 N. What work is done against gravity?

12 A yacht is sailing on a bearing of 240° with a force of 6000 N exerted on its sails by a south-easterly wind. What work is done on the yacht when it sails a distance of 1 nautical mile (n.m.)? (1 n.m. = 1852 m)

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13 A downhill skier loses 15% of the energy gained going down a 50 m long slope at an angle of 60° to the horizontal, due to friction and air resistance. What energy is gained (from gravity) if the skier's total mass (with skis, etc.) is 78 kg and the vertical force of gravity is $9.81 \times m$?

14 Example 18 Use the scalar product to prove that the diagonals of a parallelogram are perpendicular if and only if it is a rhombus.

15 Use the scalar product to prove that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.
Hint: Use $AC^2 = \mathbf{AC} \cdot \mathbf{AC}$ and express the scalar product in terms of all the sides.

# 4.08 APPLICATION OF VECTORS TO NAVIGATION

Vectors are particularly useful when constructing problems invovling navigation models. When steering a boat, the effect of the current must be taken into account. Similarly, the effect of the wind must be accounted for when determining the direction in which a plane must fly to reach its destination. In *marine navigation*, the direction in which a boat is steered is called the **heading**. The **speed through the water** is the speed of the boat relative to the water. The **speed made good** and **course made good** are the true speed and course of the boat when the effects of the current are taken into account. Taken together, the speed and course made good constitute a single vector of the true velocity.

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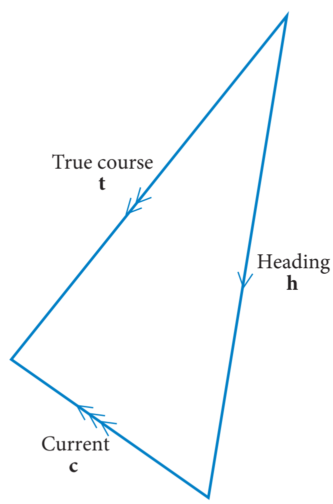
A boat is carried with the current, so the speed and course made good is the vector sum of the boat's heading and speed through the water and the current. This is illustrated in the vector triangle on the right. It is standard practice to use one arrowhead for the vector showing the course steered and the boat's speed through the water (heading), two arrowheads for the vector showing the course and speed made good (true course), and three arrowheads for the current vector.



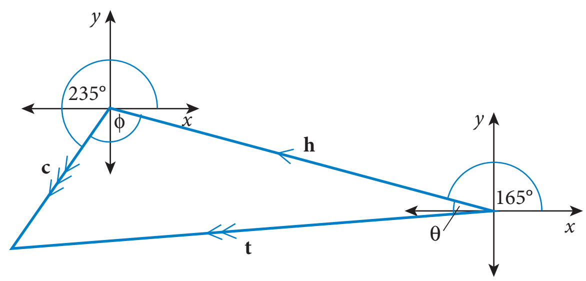

Using **h** for the heading, **c** for the current and **t** for the course made good, you get $\mathbf{t} = \mathbf{h} + \mathbf{c}$.

## Example 19

A yacht making 7 knots through the water is headed on a bearing of 285°. There is a current of 3 knots on a bearing of 215°. What is the speed and course made good?

### Solution

Start by making a rough sketch.
Take east as the **i** direction and north as the **j** direction.
The heading **h** is on a bearing of 285°.
This means that **h** is at an angle of 165° to the **i** direction.
The current **c** is at an angle of 235° to the **i** direction.




First calculate the magnitude of **t**.

Write **h** in terms of **i** and **j**.
Evaluate and round off.

$$\mathbf{h} = 7\cos(165°)\,\mathbf{i} + 7\sin(165°)\,\mathbf{j}$$
$$\approx -6.761\mathbf{i} + 1.812\mathbf{j}$$

Write **c** in terms of **i** and **j**.
Evaluate and round off.

$$\mathbf{c} = 3\cos(235°)\,\mathbf{i} + 3\sin(235°)\,\mathbf{j}$$
$$\approx -1.721\mathbf{i} - 2.457\mathbf{j}$$

Find **t**, using vector addition.

$$\mathbf{t} = \mathbf{h} + \mathbf{c}$$

Substitute for **h** and **c**.
Evaluate.

$$\approx -6.761\mathbf{i} + 1.812\mathbf{j} + (-1.721)\mathbf{i} - 2.457\mathbf{j}$$
$$= -8.482\mathbf{i} - 0.645\mathbf{j}$$

Find the magnitude of **t**.
Evaluate and round off.

$$|\mathbf{t}| = \sqrt{(-8.482)^2 + (-0.645)^2}$$
$$\approx 8.506$$

Now calculate the direction of **t**.

Use the diagram to find $\phi$.

$$\phi = 360° - 235° - (180° - 165°)$$
$$= 110°$$

Use the sine rule to find $\theta$.

$$\frac{8.506}{\sin(110°)} = \frac{3}{\sin(\theta)}$$

Rearrange and evaluate.

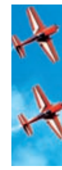
$$\theta \approx 19.4°$$

Calculate the bearing of **t**.

$$\text{Bearing of } \mathbf{t} \approx 360° - (165° - 90°) - 19.4°$$
$$= 265.6°$$

State the result.

The speed and course made good is about 8.5 knots on a bearing of 265.6°.



For Example **18**, you can use a CAS calculator to add the vectors, but you need to change the bearings measured clockwise from north into angles measured anticlockwise from the *x*-axis. When you get your answer, you need to change the angle back to a bearing.

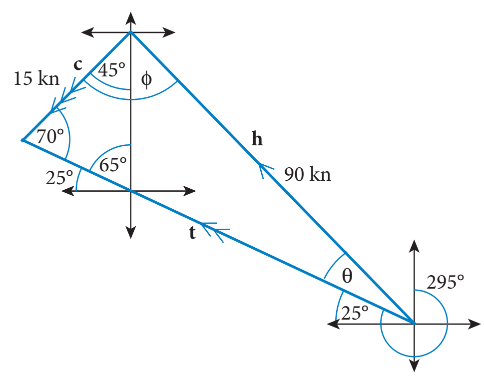
In *air navigation*, the equivalent of speed through the water is **airspeed**, shown with one arrowhead. The course made good is referred to as the **true course**, and the speed made good is called the **ground speed**, shown with two arrowheads. The air current is the **wind** and this is shown with three arrowheads. In air navigation, you normally know the wind direction and speed from meteorological data, and choose your heading in order to fly in a particular direction. In other words, **t** and **c** are known, and you wish to calculate **h**.

## Example 20

A pilot calculating a flight plan for a single-engine Cessna wants to fly on a true course of 295°. The wind is a 15-knot north-easterly and she plans to fly at an airspeed of 90 knots. What heading should she make and what will be the ground speed?

### Solution

Make a rough sketch.
A north-easterly wind blows from NE, so it blows towards SW.
You could solve this problem using vector components, but it is easier to use the vector triangle.
Use basic geometry to find the remaining angles.




Use the sine rule to find θ.

$$\frac{15}{\sin(\theta)} = \frac{90}{\sin(70°)}$$

Rearrange and evaluate.

$$\theta = 9.010...°$$

Calculate the bearing of **h**.

$$\text{Bearing of } \mathbf{h} = 295° + 9.010...° = 304.010...°$$

Calculate the third angle of the triangle.

$$\text{Third angle} = 180° - 70° - 9.010...° = 100.989...°$$

Use the sine rule to find *t*.

$$\frac{t}{\sin(100.989...°)} = \frac{90}{\sin(70°)}$$

Rearrange and evaluate.

$$t = 94.019...$$

State the result.

The pilot's heading will be about 304° and the ground speed is about 94 knots.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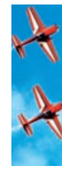
# EXERCISE 4.08 Application of vectors to navigation

## Concepts and techniques

You should use your CAS calculator wherever appropriate in this exercise.

Aerial and nautical applications



1 In marine navigation, what is the direction in which a boat is steered called?
 A bearing  B course  C heading
 D direction  E course made good

2 In air navigation, what is the *speed made good* of an aircraft called?
 A ground speed  B airspeed  C true speed
 D wind speed  E speed through the air

## Reasoning and communication

3 A boat travels 5 km NE and then 7 km at a bearing of 120°. Find its distance and direction from its starting point.

4 A strong swimmer who can swim at 6 km/h for a short period is caught in a rip 20 m from the beach. The rip is moving at 10 km/h away from the beach. Knowing safety procedures, the swimmer tries to swim across the rip, He reaches the edge of the rip after 3 minutes. How far from the shore is he?

5 An aircraft travelling at 140 knots at a bearing of 197° changes its direction to a bearing of 116° at the same speed to approach the runway from the seaward side. Find the change in velocity.

6 Example 19 Find the speed and course made good for a yacht sailing at 7 knots at 135° with a current of 2 knots at 060°.

7 Find the speed and course made good for a boat sailing at 12 knots at 200° with a current of 4 knots at 340°.

8 Example 20 The owner of a cruiser wants to sail on a true course of 120°. The current is very strong, being 5 knots towards the NW. The cruiser can make a speed of 15 knots through the water. At what heading should the owner steer the cruiser, and what will be the true speed?

9 A pilot wants to fly on a true course of 320°. The airspeed is 130 knots and wind is a steady westerly of 15 knots. Find the heading he should take and the speed over the ground of the aircraft.

10 Wind can be much stronger at high altitudes. A passenger airliner with a cruising speed of 560 km/h needs to travel 1500 km south-west at cruising height before beginning the descent for landing. At this height, the wind is an 80 km/h northerly. The plane reaches cruising height at 10:05 a.m. When will the plane commence the descent for landing?

11 A whale is moving up the east coast of Australia in a direction that is essentially at a bearing of 25°. The whale moves at a leisurely pace of about 8 knots through the water and the current up the east coast is about 2 knots northwards. What is the speed that would be observed by a whale spotter at Point Lookout?

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12 A dinghy with an outboard motor has a speed through the water of 8 knots. It is moving directly across the mouth of a creek where the incoming tide has a speed of 5 knots. Find the true speed and direction of the dinghy.

13 Find the ground speed and true course for a light plane heading 210° at an airspeed of 120 knots if the wind is a southerly at 20 knots.

14 A pilot wants to fly 600 km at a bearing of 015° in a plane capable of 95 knots. The wind is predicted to be a south-easterly at 25 knots. If she takes off at 10:30 a.m., what heading should she take, and what is her estimated time of arrival? (1 knot = 1.852 km $h^{-1}$)


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# CHAPTER SUMMARY
# APPLICATIONS OF VECTORS 4

- The **dot product** of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ with an angle of $\theta$ between them is given by:

  $\mathbf{v}_1 \cdot \mathbf{v}_2 = |\mathbf{v}_1| \times |\mathbf{v}_2| \cos(\theta)$

- The scalar product of two vectors $\mathbf{v}_1 = (a_1, a_2)$ and $\mathbf{v}_2 = (b_1, b_2)$, is given by:

  $\mathbf{v}_1 \cdot \mathbf{v}_2 = (a_1, a_2) \cdot (b_1, b_2) = a_1\, b_1 + a_2\, b_2$

- The angle between two vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ is given by:

  $$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{a_1 b_1 + a_2 b_2}{\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}}$$

- The properties of the scalar product include:

  **1** the scalar product is a real number, *not a vector*

  **2** the scalar product is commutative, namely

  $$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

  **3** the scalar product is distributive over vector addition

  $$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

  **4** $m(\mathbf{a} \cdot \mathbf{b}) = (m\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (m\mathbf{b})$, for $m$ real and vectors $\mathbf{a}$ and $\mathbf{b}$

  **5** $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$

- If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular (or **orthogonal**), then $\mathbf{a} \cdot \mathbf{b} = 0$ and vice-versa for non-zero $\mathbf{a}$ and $\mathbf{b}$.

- If $\mathbf{a}$ and $\mathbf{b}$ are parallel, then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$ or $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}||\mathbf{b}|$.

- For the unit vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$:

  $$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$$

  $$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$$

- The **projection** of $\mathbf{a}$ on $\mathbf{b}$ is a vector $\mathbf{p}$ in the direction of $\mathbf{b}$ with magnitude given by:

  $p = |\mathbf{a}| \cos(\theta)$ where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$

  or $p = \mathbf{a} \cdot \hat{\mathbf{b}}$ where $\hat{\mathbf{b}}$ is a unit vector in the direction of $\mathbf{b}$

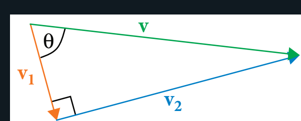
  or $p = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$

  The vector projection of $\mathbf{a}$ on $\mathbf{b}$ is given by

  $\mathbf{p} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}\hat{\mathbf{b}} = |\mathbf{a}|\cos(\theta)\hat{\mathbf{b}}$, where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$

- **Orthogonal components** of a vector are projections of the vector in perpendicular directions. Writing a given vector as a sum of orthogonal components is called **resolving** the vector into orthogonal components.

  If the angle between a vector $\mathbf{v}$ and one of the directions in which you want to resolve it is $\theta$, then the orthogonal components are given by $v_1 = v\cos(\theta)$ and $v_2 = v\sin(\theta)$.




- The work done by a force in moving an object is:

  $W = \mathbf{F} \cdot \mathbf{s}$ where $\mathbf{F}$ is the force and $\mathbf{s}$ is the displacement. If the force is in the direction of the displacement then $W = Fs$.

- The velocity of a boat through water is called the **speed through the water** and is called the **heading**. The **speed** and **course made good** (**true course**) is the boat's velocity relative to the land (Earth).
- For air navigation, speed through the air is called the **airspeed** and the speed relative to the Earth is called the **ground speed**.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4

# CHAPTER REVIEW
# APPLICATIONS OF VECTORS

## Multiple choice

1 Example 5 If $\mathbf{p} = (0, 3)$ and $\mathbf{q} = (0, -3)$, then $\mathbf{p} \cdot \mathbf{q}$ is:

A $-9$ B $-6$ C $0$ D $6$ E $9$

2 Example 12 The projection of $\mathbf{b} = -3\mathbf{i} + \mathbf{j}$ on the negative direction of the $x$-axis is:

A $0$ B $1$ C $3$ D $\sqrt{10}$ E $10$

3 Example 11 The projection of $\mathbf{u} = (5, 30°)$ in the direction 210° is:

A $-10$ B $-5$ C $0$ D $5$ E $10$

4 Example 11 The projection of $\mathbf{a} = (3, 160°)$ on $\mathbf{b} = (8, 70°)$ is:

A $-8$ B $-3$ C $0$ D $3$ E $8$

5 Example 12 The projection of $4\mathbf{i} + 2\mathbf{j}$ on $\frac{1}{2}\mathbf{i} - \mathbf{j}$ is:

A $-1$ B $0$ C $\frac{1}{2}$ D $2$ E $4$

6 Example 13 A plane is flying toward the south-east. Which of the following wind velocity vectors increases the plane's speed the most?

A $\mathbf{w}_1 = 6\mathbf{i} - 3\mathbf{j}$ B $\mathbf{w}_2 = 6\mathbf{j}$ C $\mathbf{w}_3 = 6\mathbf{i} + \mathbf{j}$

D $\mathbf{w}_4 = -6\mathbf{i} - 6\mathbf{j}$ E $\mathbf{w}_5 = -6\mathbf{i} + 6\mathbf{j}$

7 Example 13 For the plane described in question **6**, which of the following wind velocity vectors slows down the plane the most?

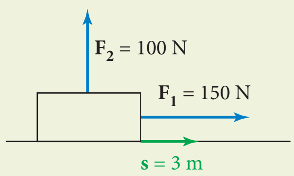
A $\mathbf{w}_1 = 6\mathbf{i} - 3\mathbf{j}$ B $\mathbf{w}_2 = 6\mathbf{j}$ C $\mathbf{w}_3 = 6\mathbf{i} + \mathbf{j}$

D $\mathbf{w}_4 = -6\mathbf{i} - 6\mathbf{j}$ E $\mathbf{w}_5 = -6\mathbf{i} + 6\mathbf{j}$

8 Example 16 An object is subjected to two forces, as shown in the diagram on the right. If the object has a displacement of 3 m, then the work done by the forces is:

A 0 B 200 J C 250 J

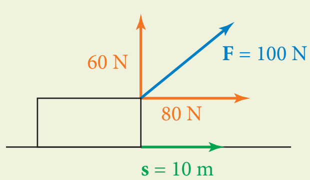
D 450 J E 750 J




9 Example 17 An object is subjected to a force of 100 N, as shown in the diagram on the right. The components of the force parallel and perpendicular to the displacement (**s**) are 80 N and 60 N respectively. The work done by **F** is:

A 600 J B 800 J C 1000 J

D 1400 J E 2400 J



## Short answer

10 Example 1 The angle between vectors **a** and **b** is $\theta$. Find $\mathbf{a} \cdot \mathbf{b}$ (correct to 2 decimal places if necessary) when:

a $|\mathbf{a}| = 3$, $|\mathbf{b}| = 7$ and $\theta = 12°$

b $|\mathbf{a}| = 9$, $|\mathbf{b}| = 5$ and $\theta = 71°$

11 Example 4 Use the geometric definition to calculate the dot product of each of the following pairs of vectors.

a $3\mathbf{i} - 5\mathbf{j}$ and $-4\mathbf{i} + 4\mathbf{j}$

b $-3\mathbf{i} - 4\mathbf{j}$ and $2\mathbf{i} + 5\mathbf{j}$

12 Example 3 The dot product of $\mathbf{d} = (1, 7)$ and $\mathbf{e} = (6, 2)$ is 20. Calculate the angle between **d** and **e**.

13 If $\mathbf{m} = (8, 32°)$ and $\mathbf{n} = (12, 325°)$, calculate the value of $\mathbf{m} \cdot \mathbf{n}$.

14 Example 4 Calculate the dot product of (1, 5) and (3, 7) using both the algebraic and geometric definitions of the dot product.

15 Example 4 Calculate the scalar product of each of the following pairs of vectors.

a $-3\mathbf{i} + 10\mathbf{j}$ and $5\mathbf{i} - 4\mathbf{j}$

b $(-2, -3)$ and $(7, 9)$

c $\begin{bmatrix} 21 \\ 12 \end{bmatrix}$ and $\begin{bmatrix} -6 \\ 11 \end{bmatrix}$

16 Example 5 Calculate the scalar product of the following pair of vectors, correct to two decimal places if necessary.

a $(5, 2)$ and $(7, 80°)$

b $\left(12, \frac{\pi}{3}\right)$ and $(4, -7)$

17 Example 6 Calculate the angle between the following pairs of vectors.

a $(1, 2)$ and $(6, 8)$

b $(3, -4)$ and $(-2, 3)$

18 Example 7 Use the pair of vectors $(3, -7)$ and $(4, 2)$ to demonstrate that the scalar product is commutative (i.e. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$).

19 Example 9 Show that the following pairs of vectors are perpendicular.

a $(3, 0)$ and $(0, -6)$

b $(5, 2)$ and $(2, -5)$

20 Example 10 Show that the following pairs of vectors are parallel.

a $(9, 12)$ and $\left(1, \frac{4}{3}\right)$

b $(-5, -2)$ and $\left(1, \frac{2}{5}\right)$

21 Example 10 Determine if the following pairs of vectors are perpendicular, parallel or neither.

a $(3, 2)$ and $(-2, 3)$

b $(5, -1)$ and $(2, 3)$

c $\left(-4, \frac{3}{2}\right)$ and $(8, -3)$

22 Example 11 Find the projection in the direction 210° of a vector of magnitude 50 and direction 280°.

23 Example 12 Find the projection of $\mathbf{p} = (2, 7)$ on $\mathbf{q} = (5, 12)$.

24 Example 15 Given vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$ and force vector $\mathbf{F} = -2\mathbf{i} + 4\mathbf{j}$, calculate:

a the component of **F** parallel to **v**

b the component of **F** perpendicular to **v**

25 Example 13 A force of 200 N acts in the direction N 25° E. Resolve the force into components in the north and east directions.



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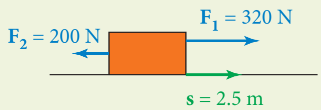
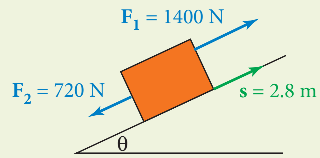
26 Example 16 Calculate the work done in each of the following situations.

a




b




## Application

27 Prove that $|\mathbf{m}| = \sqrt{\mathbf{m} \cdot \mathbf{m}}$, where $\mathbf{m} = (m_1, m_2)$.

28 Consider the points $A(-6, 7)$, $B(-2, 3)$ and $C(3, 4)$. Calculate the angle between **BA** and **BC**.

29 Two ice-skaters are helping a beginner to get started. The ice-skater on the left is exerting a force of 15 N at an angle of 30° from the forward direction and to the left. The skater on the right is exerting a force of 20 N at an angle of 40° from the forward direction and to the right. What is the total force acting on the beginner?

30 A car going up a hill inclined at 10° has a weight of 8500 N acting vertically downwards. Resolve the weight into components parallel and perpendicular to the slope.

31 Four people are attempting to pull a tree stump out of the ground. The first person is pulling at an angle of 30° to the horizontal with a force of 200 N. The second person is 60° clockwise around the stump from the first and is pulling at an angle of 56° to the horizontal with a force of 155 N. The third person is 75° anticlockwise from the first and is pulling with a force of 300 N at an angle of 48° to the horizontal. The fourth person is directly opposite the first and is pulling with a force of 300 N at an angle of 45° to the horizontal. Find the total force acting on the stump, and the lifting force.

32 Use the scalar product to find the amount of work that is done when a force of 200 N moves its point of application 20 m at an angle of 36° to the force.

Practice quiz

33 Find the heading needed and speed made good for a true course of 135° in a boat capable of 8 knots in a current of 2 knots in the direction 045°.

34 Find the heading required to fly a plane with an airspeed of 105 knots on a true course of 225° with a southwesterly wind of 10 knots. Find the ground speed.

35 Use the scalar product to prove the diagonals of a kite are perpendicular.